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Periodic boundary value problem for the first order impulsive functional differential equations ☆

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Abstract

This paper is concerned with the existence of extreme solutions of the periodic boundary value problem for a class of first order impulsive functional differential equations. We introduce new concept of lower and upper solutions and present that the method of lower and upper solutions coupled with monotone iterative technique is still valid. Meanwhile, we extend previous results.

MSC: Primary 34B37

Keywords: Impulsive differential equation; Monotone iterative technique; Periodic boundary value problem

1. Introduction

The theory of impulsive differential equations has become an important aspect of differential equations (see [7,1]). Periodic boundary value problem (PBVP for short) for impulsive differential equations has drawn much attention. In papers [10,5] and the references in them, PBVP for first order impulsive functional differential has been studied. For second order impulsive differential equations, see [2,4]. Recently, Hristova and Roberts [6] considered PBVP for first order impulsive differential equations with "supremum" and Nieto and Rordriguez [9,8] introduced a new concept of lower and upper solutions for the first order functional differential equation

$$\begin{cases} u'(t) = g(t, u(t), u(\theta(t))), \quad 0 \leqslant \theta(t) \leqslant t, \quad t \in J = [0, T], \\ u(0) = u(T). \end{cases}$$

In a recent paper [3], the authors studied the PBVP for the first order impulsive functional differential equation

$$\begin{cases} u'(t) = g(t, u(t), u(\theta(t))), \quad t \neq t_k, \quad t \in J = [0, T], \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{1.1}$$

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where $g \in C(J \times R^2, R)$, $0 \leqslant \theta(t) \leqslant t$, $t \in J$, $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, $I_k \in C(R, R)$. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k = 1, 2, \ldots, p$.

Motivated by [6,9,8,3], we shall study the PBVP (1.1) and establish a new comparison p principle under a similar definition of lower and upper solutions.

We note that if $\theta(t) = t$, then Eq. (1.1) is an ordinary differential equation with impulsive whose PBVP has been studied in [7].

This paper is organized as follows: in Section 2, we establish a new comparison principle. In Section 3, we first introduce new concept of lower and upper solutions, and then give a different proof for the existence theorem related to a linear problem associated to Eq. (1.1). In Section 4, by using the method of upper and lower solutions and monotone iterative technique, we obtain the existence of extreme solutions for PBVP (1.1).

2. Some lemmas

Let $J^- = J \backslash \{t_1, t_2, \ldots, t_p\}$, $\mathrm{PC}(J, R) = \{u : J \to R; u(t)$ continuous everywhere except for some t_k at which $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k), k = 1, 2, \ldots, p\}$, $\mathrm{PC}'(J, R) = \{u : J \to R; u'(t)$ is continuous everywhere except for some t_k at which $u'(t_k^+)$ and $u'(t_k^-)$ exist and $u'(t_k^-) = u'(t_k), k = 1, 2, \ldots, p\}$. Let $E = \{u | u \in \mathrm{PC}(J, R) \cap \mathrm{PC}'(J, R)\}$ with norm $\|u\|_E = \sup_{t \in J} |u(t)|$, then E is a Banach space. A function $u \in E$ is called a solution of PBVP (1.1) if it satisfies (1.1).

We now present a comparison result.

Lemma 2.1. *Let $u \in E$, $M > 0$, $N \geqslant 0$, $0 \leqslant L_k < 1$, such that*

$$u'(t) + Mu(t) + Nu(\theta(t)) \leqslant 0, \quad t \in J^-, \tag{2.1}$$

$$\Delta u(t_k) \leqslant -L_k u(t_k), \quad k = 1, 2, \ldots, p, \tag{2.2}$$

$$u(0) \leqslant 0, \tag{2.3}$$

$$N \int_0^T \mathrm{e}^{M(t-\theta(t))}\, \mathrm{d}t + \sum_{k=1}^{p} L_k \leqslant 1. \tag{2.4}$$

Then $u \leqslant 0$ on J.

Proof. Let $v(t) = \mathrm{e}^{Mt} u(t)$, $t \in [0, T]$. Then, using (2.1),

$$v'(t) = M\mathrm{e}^{Mt} u(t) + \mathrm{e}^{Mt} u'(t) \leqslant -N\mathrm{e}^{Mt} u(\theta(t)), \quad t \in J^-,$$

or, equivalently,

$$v'(t) \leqslant -N\mathrm{e}^{M(t-\theta(t))} v(\theta(t)), \quad t \in J^-. \tag{2.5}$$

The functions u and v have the same sign, so we need only prove that $v \leqslant 0$ on J. If this was false, there would exist $t^* \in J$ such that $v(t^*) > 0$. Since $v(0) = u(0) \leqslant 0$, then $t^* \in (0, T]$. Let $t_* \in [0, t^*)$ such that $v(t_*) = \inf_{t \in [0,\ t^*]} v(t) \leqslant 0$. Assume that $t^* \in [t_i, t_{i+1})$ and $t_* \in [t_j, t_{j+1})$, where $i, j \in \{0, \ldots, p-1\}$, so $j \leqslant i$. Now, if we integrate expression (2.5) between t_* and t^*, we obtain that

$$\begin{aligned} v(t^*) - v(t_*) &\leqslant -N \int_{t_*}^{t^*} \mathrm{e}^{M(t-\theta(t))} v(\theta(t))\, \mathrm{d}t + \sum_{k=i+1}^{j} \Delta v(t_k) \\ &\leqslant -v(t_*) N \int_{t_*}^{t^*} \mathrm{e}^{M(t-\theta(t))}\, \mathrm{d}t - v(t_*) \sum_{k=i+1}^{j} L_k \\ &\leqslant -v(t_*) \left[N \int_0^T \mathrm{e}^{M(t-\theta(t))}\, \mathrm{d}t + \sum_{k=1}^{p} L_k \right] \leqslant -v(t_*), \end{aligned}$$

where we have taken into account that $0 \leqslant \theta(t) \leqslant t$, for $t \in J$ and condition (2.4). Thus, we obtain $v(t^*) \leqslant 0$, which is absurd. This proves that $v \leqslant 0$ on J, and so, $u \leqslant 0$ on J. □

Corollary 2.1. *Let the function $u \in C'([0,T],R)$ be such that inequalities* (2.1), (2.3) *hold, where the constants N and M satisfy*

$$N \int_0^T \mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t \leqslant 1.$$

Then $u(t) \leqslant 0$ for $t \in [0,T]$.

Lemma 2.2. *Let the function $u \in E$ satisfy inequalities* (2.1), (2.2) *and $u(0) \leqslant u(T)$, where the constants $M > 0$, $N \geqslant 0$ and $0 \leqslant L_k < 1$, $(k = 1, 2, \ldots, p)$ satisfy inequality* (2.4). *Then the function $u(t)$ is nonpositive in the interval* $[0,T]$.

Proof. If $u \geqslant 0$ on J, then $u'(t) \leqslant 0$ on J, so u is nonincreasing function. This fact joint to $u(0) \leqslant u(T)$ produces that u is a constant function, so that $u' \equiv 0$, and also $u \equiv 0$.

Thus, we can consider that u takes some negative value. The proof consists on demonstrating that $u(0) \leqslant 0$ so that we could apply Lemma 2.1 and affirm that $u \leqslant 0$. If $u(0) > 0$, also $u(T) > 0$, and considering again the function v defined by $v(t) = \mathrm{e}^{Mt}u(t)$, $t \in [0,T]$, we obtain that $v(0) > 0$, $v(T) > 0$ and $v(t_*) = \inf_{[0,T]} v < 0$, with $t_* \in (0,T)$. And assume $t_* \in [t_l, t_{l+1})$. The integration of (2.5) between t_* and T yields

$$\begin{aligned}
-v(t_*) < v(T) - v(t_*) &\leqslant -N \int_{t_*}^T v(\theta(t))\mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t + \sum_{k=l+1}^{p} \Delta v(t_k) \\
&\leqslant -Nv(t_*) \int_{t_*}^T \mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t - \sum_{k=l+1}^{p} L_k v(t_k) \\
&\leqslant -v(t_*) \left[N \int_0^T \mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t + \sum_{k=1}^{p} L_k \right] \\
&\leqslant -v(t_*),
\end{aligned}$$

which is absurd. Then $u(0) \leqslant 0$ and the conclusion follows. □

Lemma 2.3. *Assume that $u \in E$ satisfies*

$$u'(t) + Mu(t) + Nu(\theta(t)) + \frac{Mt + N\theta(t) + 1}{T}[u(0) - u(T)] \leqslant 0, \quad t \in J^-, \tag{2.6}$$

$$\Delta u(t_k) \leqslant -L_k u(t_k) - \frac{L_k t_k}{T}[u(0) - u(T)], \quad k = 1, 2, \ldots, p, \tag{2.7}$$

$$u(0) > u(T), \tag{2.8}$$

where constants $M > 0$, $N \geqslant 0$, $0 \leqslant L_k < 1$ $(k = 1, 2, \ldots, p)$, and they satisfy (2.4). *Then $u(t) \leqslant 0$ for all $t \in J$.*

Proof. Setting

$$x(t) = u(t) + \frac{t}{T}(u(0) - u(T)), \quad t \in J,$$

then $x(t)\geqslant u(t)$, and for $t\neq t_k,\ t\in J$,

$$\begin{aligned}x'(t)+Mx(t)+Nx(\theta(t)) &= u'(t)+\frac{1}{T}[u(0)-u(T)]+Mu(t)\\ &\quad+\frac{Mt}{T}[u(0)-u(T)]+Nu(\theta(t))+\frac{N\theta(t)}{T}[u(0)-u(T)]\\ &= u'(t)+Mu'(t)+Nu(\theta(t))\\ &\quad+\frac{Mt+N\theta(t)+1}{T}[u(0)-u(T)]\leqslant 0.\end{aligned}$$

By direct calculus, we have $\Delta x(t_k)=\Delta u(t_k)\leqslant -L_k u(t_k)-(L_k t_k/T)[u(0)-u(T)]=-L_k x(t_k)$, $x(0)=u(0)=x(T)$. By Lemma 2.2, $x(t)\leqslant 0$ on J, which implies that $u(t)\leqslant 0$. Hence we complete the proof. □

3. Existence of solutions for the linear problem

Definition 3.1. A function $\alpha\in E$ is called a lower solution of PBVP (1.1) if

$$\begin{cases}\alpha'(t)\leqslant g(t,\alpha(t),\alpha(\theta(t)))-a(t), & t\neq t_k,\ t\in J,\\ \Delta\alpha(t_k)\leqslant I_k(\alpha(t_k))-l_k, & k=1,2,\ldots,p,\end{cases}$$

where

$$a(t)=\begin{cases}0, & \alpha(0)\leqslant\alpha(T),\\ \dfrac{Mt+N\theta(t)+1}{T}[\alpha(0)-\alpha(T)], & \alpha(0)>\alpha(T),\end{cases}$$

$$l_k=\begin{cases}0, & \alpha(0)\leqslant\alpha(T),\\ \dfrac{L_k t_k}{T}[\alpha(0)-\alpha(T)], & \alpha(0)>\alpha(T).\end{cases}$$

Definition 3.2. A function $\beta\in E$ is called an upper solution of PBVP (1.1) if

$$\begin{cases}\beta'(t)\geqslant g(t,\beta(t),\beta(\theta(t)))+b(t), & t\neq t_k,\ t\in J,\\ \Delta\beta(t_k)\geqslant I_k(\beta(t_k))+l_k^*, & k=1,2,\ldots,p,\end{cases}$$

where

$$b(t)=\begin{cases}0, & \beta(0)\geqslant\beta(T),\\ \dfrac{Mt+N\theta(t)+1}{T}[\beta(T)-\beta(0)], & \beta(0)<\beta(T),\end{cases}$$

$$l_k^*=\begin{cases}0, & \beta(0)\geqslant\beta(T),\\ \dfrac{L_k t_k}{T}[\beta(T)-\beta(0)], & \beta(0)<\beta(T).\end{cases}$$

Remarks. The definition of classical lower and upper solutions makes reference to the case $\alpha(0)\leqslant\alpha(T),\ \beta(0)\geqslant\beta(T)$, hence we enlarge some results in [10,5].

Theorem 3.1. *Let $\sigma\in C(J),\ M>0,\ N\geqslant 0,\ 0\leqslant L_k<1$ and γ_k $(k=1,2,\ldots,p)$ are constants. Consider the problem*

$$\begin{cases}u'(t)+Mu(t)+Nu(\theta(t))=\sigma(t), & t\neq t_k,\ t\in J,\\ \Delta u(t_k)=-L_k u(t_k)+\gamma_k, & t=t_k,\ k=1,2,\ldots,p,\\ u(0)=u(T).\end{cases}\tag{3.1}$$

Suppose that there exist $\alpha, \beta \in E$ *such that*

(h_1) $\alpha \leqslant \beta$ *on* J.

(h_2)

$$\begin{cases} \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \leqslant \sigma(t) - a(t), \\ \Delta\alpha(t_k) \leqslant -L_k\alpha(t_k) - l_k + \gamma_k, \end{cases}$$

where

$$a(t) = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}[\alpha(0) - \alpha(T)] & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$l_k = \begin{cases} 0 & \text{if } \alpha(0) \leqslant \alpha(T), \\ \dfrac{L_k t_k}{T}[\alpha(0) - \alpha(T)] & \text{if } \alpha(0) > \alpha(T). \end{cases}$$

(h_3)

$$\begin{cases} \beta'(t) + M\beta(t) + N\beta(\theta(t)) \geqslant \sigma(t) + b(t), \\ \Delta\beta(t_k) \geqslant -L_k\beta(t_k) + l_k^* + \gamma_k, \end{cases}$$

where

$$b(t) = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{Mt + N\theta(t) + 1}{T}[\beta(T) - \beta(0)] & \text{if } \beta(0) < \beta(T), \end{cases}$$

$$l_k^* = \begin{cases} 0 & \text{if } \beta(0) \geqslant \beta(T), \\ \dfrac{L_k t_k}{T}[\beta(T) - \beta(0)] & \text{if } \beta(0) < \beta(T). \end{cases}$$

(h_4)

$$N\int_0^T \mathrm{e}^{M(t-\theta(t))}\,\mathrm{d}t + \sum_{k=1}^{p} L_k \leqslant 1.$$

Then, there exists a unique solution u *for problem* (3.1). *Moreover,* $u \in [\alpha, \beta]$.

Proof. We first prove the uniqueness of solution for this problem. If u_1, u_2 are solutions of (3.1), set $v_1 = u_1 - u_2$ and $v_2 = u_2 - u_1$. Thus,

$$v_1(0) = v_1(T), \quad \Delta v_1(t_k) = -L_k v_1(t_k), \quad k = 1, 2, \ldots, p,$$

$$v_1(t) + Mv_1(t) + Nv_1(\theta(t)) = 0, \quad t \in J^-,$$

and

$$v_2(0) = v_2(T), \quad \Delta v_2(t_k) = -L_k v_2(t_k), \quad k = 1, 2, \ldots, p,$$

$$v_2(t) + Mv_2(t) + Nv_2(\theta(t)) = 0, \quad t \in J^-.$$

By Lemma 2.2, we have that $v_1 = u_1 - u_2 \leqslant 0$ and $v_2 = u_2 - u_1 \leqslant 0$, and hence $u_1 = u_2$.

Now, we show that if u is a solution to (3.1), then $\alpha \leqslant u \leqslant \beta$. Define $m_1 = \alpha - u$ and $m_2 = u - \beta$. We can write that If $m_1(0) \leqslant m_1(T)$, then

$$\begin{cases} m_1(t) + Mm_1(t) + Nm_1(\theta(t)) \leqslant 0, \\ \Delta m_1(t_k) \leqslant -L_k m_1(t_k), \end{cases}$$

if $m_1(0) > m_1(T)$, then

$$\begin{cases} m_1(t) + Mm_1(t) + Nm_1(\theta(t)) + \dfrac{Mt + N\theta(t) + 1}{T}[m_1(0) - m_1(T)] \leqslant 0, \\ \Delta m_1(t_k) \leqslant -L_k m_1(t_k) - \dfrac{L_k t_k}{T}[m_1(0) - m_1(T)]. \end{cases}$$

If $m_2(0) \leqslant m_2(T)$, then

$$\begin{cases} m_2(t) + Mm_2(t) + Nm_2(\theta(t)) \leqslant 0, \\ \Delta m_2(t_k) \leqslant -L_k m_2(t_k), \end{cases}$$

if $m_2(0) > m_2(T)$, then

$$\begin{cases} m_2(t) + Mm_2(t) + Nm_2(\theta(t)) + \dfrac{Mt + N\theta(t) + 1}{T}[m_2(0) - m_2(T)] \leqslant 0, \\ \Delta m_2(t_k) \leqslant -L_k m_2(t_k) - \dfrac{L_k t_k}{T}[m_2(0) - m_2(T)]. \end{cases}$$

Now, Lemmas 2.2 and 2.3 allow to assure that $m_1 = \alpha - u \leqslant 0$ and $m_2 = u - \beta \leqslant 0$.

To prove the existence of solution, we consider the functions

$$\bar{\alpha}(t) = \begin{cases} \alpha(t) & \text{if } \alpha(0) \leqslant \alpha(T), \\ \alpha(t) + \dfrac{t}{T}(\alpha(0) - \alpha(T)) & \text{if } \alpha(0) > \alpha(T) \end{cases} \tag{3.2}$$

and

$$\bar{\beta}(t) = \begin{cases} \beta(t) & \text{if } \beta(0) \geqslant \beta(T), \\ \beta(t) - \dfrac{t}{T}(\beta(T) - \beta(0)) & \text{if } \beta(0) < \beta(T). \end{cases} \tag{3.3}$$

It is evident that $\alpha \leqslant \bar{\alpha}$ and $\bar{\beta} \leqslant \beta$ on J. Also, $\bar{\alpha}(0) = \alpha(0) \leqslant \bar{\alpha}(T)$ and $\bar{\beta}(0) = \beta(0) \geqslant \bar{\beta}(T)$. Note that, if $\alpha(0) > \alpha(T)$, $\bar{\alpha}$ is T-periodic, and the same for $\bar{\beta}$, if $\beta(0) < \beta(T)$.

We can check that $\bar{\alpha}$ and $\bar{\beta}$ are classical lower and upper solutions, respectively, for problem (3.1) and that $\bar{\alpha} \leqslant \bar{\beta}$, so that $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$. Indeed,

$$\begin{aligned} &\bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) \\ &\quad = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \leqslant \sigma(t), \quad t \in J \text{ if } \alpha(0) \leqslant \alpha(T), \\ &\Delta\bar{\alpha}(t_k) = \Delta\alpha(t_k) \leqslant -L_k\bar{\alpha}(t_k) + \gamma_k \quad \text{if } \alpha(0) \leqslant \alpha(T); \\ &\bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\qquad\qquad + \frac{Mt + N\theta(t) + 1}{T}[\alpha(0) - \alpha(T)] \leqslant \sigma(t), \quad t \in J \text{ if } \alpha(0) > \alpha(T), \\ &\Delta\bar{\alpha}(t_k) = \Delta\alpha(t_k) \leqslant -L_k\alpha(t_k) - l_k + \gamma_k \\ &\qquad = -L_k\alpha(t_k) + \frac{L_k t_k}{T}[\alpha(0) - \alpha(T)] + \gamma_k \\ &\qquad = -L_k\bar{\alpha}(t_k) + \gamma_k \quad \text{if } \alpha(0) > \alpha(T). \end{aligned}$$

And

$$\begin{aligned} &\bar{\beta}'(t) + M\bar{\beta}(t) + N\bar{\beta}(\theta(t)) \\ &\quad = \beta'(t) + M\beta(t) + N\beta(\theta(t)) \geqslant \sigma(t), \quad t \in J \text{ if } \beta(0) \geqslant \beta(T), \\ &\Delta\bar{\beta}(t_k) = \Delta\beta(t_k) \geqslant -L_k\bar{\beta}(t_k) + \gamma_k \quad \text{if } \beta(0) \geqslant \beta(T); \end{aligned}$$

$$\bar{\beta}'(t) + M\bar{\beta}(t) + N\bar{\beta}(\theta(t)) = \beta'(t) + M\beta(t) + N\beta(\theta(t))$$
$$- \frac{Mt + N\theta(t) + 1}{T}[\beta(T) - \beta(0)] \geqslant \sigma(t), \ t \in J, \quad \text{if } \beta(0) < \beta(T),$$

$$\Delta\bar{\beta}(t_k) = \Delta\beta(t_k) \geqslant -L_k\beta(t_k) + l_k^* + \gamma_k$$
$$= -L_k\beta(t_k) + \frac{L_k t_k}{T}[\beta(T) - \beta(0)] + \gamma_k$$
$$= -L_k\bar{\beta}(t_k) + \gamma_k \quad \text{if } \beta(0) < \beta(T).$$

Thus, $\bar{\alpha}$ is a classical lower solution and $\bar{\beta}$ is a classical upper solution for (3.1). Now, consider the function $m = \bar{\alpha} - \bar{\beta} \in E$. It is easy to prove that

$$m'(t) + Mm(t) + Nm(\theta(t))$$
$$= \bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) - \bar{\beta}'(t) - M\bar{\beta}(t) - N\bar{\beta}(\theta(t))$$
$$\leqslant \sigma(t) - \sigma(t) = 0, \quad t \in J^-,$$

$$\Delta m(t_k) = \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k)$$
$$\leqslant -L_k\bar{\alpha}(t_k) + \gamma_k + L_k\bar{\beta}(t_k) - \gamma_k$$
$$= -L_k m(t_k), \quad k = 1, 2, \ldots, p.$$

Also, $m(0) = \bar{\alpha}(0) - \bar{\beta}(0) = \alpha(0) - \beta(0) \leqslant 0$. Using Lemma 2.1, we obtain that $m \leqslant 0$ on J or, equivalently, $\bar{\alpha} \leqslant \bar{\beta}$ on J.

In the following, we consider the existence of solution for the equation

$$u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \quad t \in J^-, \tag{3.4}$$

$$\Delta u(t_k) = -L_k u(t_k) + \gamma_k, \quad k = 1, 2, \ldots, p, \tag{3.5}$$

$$u(0) = u(T). \tag{3.6}$$

Without loss of generality, we assume $p = 1$. Let $u_0 \in [\bar{\alpha}(0), \bar{\beta}(0)]$ be arbitrary, we first consider the linear equation

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), & t \in [0, t_1], \\ u(0) = u_0. \end{cases} \tag{3.7}$$

The initial value problem (3.7) has a unique solution $X_1(t, u_0)$ for $t \in [0, t_1]$. We shall prove that the function $X_1(t, u_0)$ satisfies the inequalities $\bar{\alpha}(t) \leqslant X_1(t, u_0) \leqslant \bar{\beta}(t)$ for $t \in [0, t_1]$.

Now, we consider the function $m(t) = \bar{\alpha}(t) - X_1(t, u_0)$. The function $m(t)$ satisfies

$$\begin{cases} m'(t) + Mm(t) + Nm(\theta(t)) \leqslant 0 \\ m(0) = \bar{\alpha}(0) - u_0 \leqslant 0, \end{cases}$$

and according to Corollary 2.1, the function $m(t)$ is nonpositive, i.e., $\bar{\alpha}(t) \leqslant X_1(t, u_0)$ for $t \in [0, t_1]$.

In a similar way, it can be proved that the inequality $X_1(t, u_0) \leqslant \bar{\beta}(t)$ holds for $t \in [0, t_1]$.

Define $y_0 = (1 - L_1)X_1(t_1; u_0) + \gamma_1$, consider

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), & t \in [t_1, T], \\ u(t_1) = y_0. \end{cases} \tag{3.8}$$

This initial value problem has a unique solution $X_2(t; y_0)$ for $t \in [t_1, T]$. By Corollary 2.1, we can prove that $X_2(t; y_0)$ satisfies the inequalities $\bar{\alpha} \leqslant X_2(t; y_0) \leqslant \bar{\beta}(t)$ for $t \in [t_1, T]$. We define the function $X(t; u_0) = X_1(t, u_0)$ for $t \in [0, t_1]$ and $X(t; u_0) = X_2(t; y_0)$ for $t \in (t_1, T]$. The function $X(t; u_0)$ is a solution of the linear impulsive differential (3.4), (3.5) with an initial condition $u(0) = u_0$.

Now we prove that there exists a point $u_0 \in [\bar{\alpha}(0), \bar{\beta}(0)]$ such that the corresponding solution $X(t; u_0)$ satisfies the periodic boundary condition (3.6).

We consider the following two cases.

Case 1: Suppose the equality $\bar{\alpha}(0) = \bar{\beta}(0)$ holds. Then it follows that the inequalities

$$u_0 = \bar{\alpha}(0) \leqslant \bar{\alpha}(T) \leqslant X(T; u_0) \leqslant \bar{\beta}(T) \leqslant \bar{\beta}(0) = u_0$$

hold. Therefore $X(T; u_0) = u_0 = X(0; u_0)$ holds.

Case 2: Suppose $\bar{\alpha}(0) < \bar{\beta}(0)$ holds. We assume the contrary, i.e., for every point $u_0 \in [\bar{\alpha}(0), \bar{\beta}(0)]$ the inequality

$$X(0; u_0) \neq X(T; u_0) \tag{3.9}$$

holds. From inequality (3.9) and the properties of the function $\bar{\alpha}$ and $\bar{\beta}$, it follows that $X(0; \bar{\alpha}(0)) < X(T; \bar{\alpha}(0))$ and $X(0; \bar{\beta}(0)) > X(T; \bar{\beta}(0))$.

We will prove that there exists a constant δ, $0 < \delta < \bar{\beta}(0) - \bar{\alpha}(0)$, such that for $0 \leqslant \bar{\beta}(0) - z < \delta$ the inequality

$$X(0; z) > X(T; z) \tag{3.10}$$

holds.

We assume the contrary, i.e., there exists a sequence of points $\{z_n\}_0^\infty$ such that $0 \leqslant \bar{\beta}(0) - z_n < \delta$ and the solution $X(t; z_n)$ satisfies the inequality

$$X(0; z_n) < X(T; z_n). \tag{3.11}$$

The function $X(t; z_n)$ satisfies the equality

$$X(t; z_n) = z_n - \int_0^t MX_1(s; z_n)\,\mathrm{d}s - \int_0^t NX_1(\theta(s); z_n)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}s \quad \text{for } t \in [0, t_1],$$

and

$$X(t; z_n) = r_n - \int_{t_1}^t MX_2(s; r_n)\,\mathrm{d}s - \int_{t_1}^t NX_2(\theta(s); z_n)\,\mathrm{d}s + \int_{t_1}^t \sigma(s)\,\mathrm{d}s \quad \text{for } t \in [t_1, t],$$

where $r_n = (1 - L_1)X_1(t_1; z_n) + \gamma_1$.

From the equalities above and the conditions of Theorem 3.1, it follows that the sequence $\{X(t; z_n)\}_0^\infty$ is uniformly bounded and completely continuous in the interval $[0, T]$ and therefore there exists an uniformly convergent subsequence of sequence $\{X(t; z_n)\}_0^\infty$. Taking the limit as $n \to \infty$, we find that the limit of the convergent sequence coincides with the function $X(t; \bar{\beta}(0))$.

Now, from inequalities (3.9) and (3.11) it follows that the function $X(t; \bar{\beta}(0))$ satisfies $X(0; \bar{\beta}(0)) < X(T; \bar{\beta}(0))$, which is a contradiction. Thus our assumption is not true, i.e., there exists a number δ with the above properties.

We denote $\delta^* = \sup\{\delta : 0 < \delta < \bar{\beta}(0) - \bar{\alpha}(0)$ such that for $0 < \bar{\beta}(0) - z < \delta$ the solution $X(t; z)$ satisfies (3.10). Then the inequalities $0 < \delta^* < \bar{\alpha}(0) - \bar{\beta}(0)$ and $X(0; z) > X(T; z)$ for $0 \leqslant \bar{\beta}(0) - z < \delta^*$ hold. From the definition of the number δ^*, it follows that there exists a sequence of points $\{x_n\}$ such that $x_n \in (\bar{\alpha}(0), \bar{\beta}(0) - \delta^*)$, $x_n \to \bar{\beta}(0) - \delta^*$ for $n \to \infty$ and the inequality $X(0; x_n) < X(T; x_n)$ holds. Taking the limit as $n \to \infty$, we obtain that

$$X(0; \bar{\beta}(0) - \delta^*) < X(0; \bar{\beta}(T) - \delta^*), \tag{3.12}$$

where $\lim_{n\to\infty} X(t; x_n) = X(t; \bar{\beta}(0) - \delta^*)$.

The function $X(t; \bar{\beta}(0) - \delta^*)$ is a solution of the following differential equation with initial condition $u(0) = \bar{\beta}(0) - \delta^*$,

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), \\ \Delta u(t_k) = -L_k u(t_k) + \gamma_k. \end{cases}$$

Inequality (3.9) contradicts the choice of the number δ^* and inequality (3.10). The contradiction obtained shows that our assumption is not true, i.e., there exists a point $u_0 \in [\bar{\alpha}(0), \bar{\beta}(0)]$ such that the corresponding solution $X(t; u_0)$ of (3.4) and (3.5) satisfies the periodic boundary condition (3.6). □

4. Main results

We will now give a procedure for constructing two sequences of functions that are respectively monotone increasing and monotone decreasing which converge to the extremal solutions of the PBVP (1.1).

Theorem 4.1. *Suppose the following conditions are fulfilled*:

(H_1) *The function* $\alpha, \beta \in E$ *are lower and upper solutions*, *respectively*, *of the PBVP* (1.1) *such that* $\alpha \leqslant \beta$ *for* $t \in [0, T]$.

(H_2) *The function* $g \in C([0, T] \times R \times R, R)$ *and for* $\alpha(t) \leqslant x_1(t) \leqslant x_2(t) \leqslant \beta(t)$, $\alpha(\theta(t)) \leqslant y_1(\theta(t)) \leqslant y_2(\theta(t)) \leqslant \beta(\theta(t))$ *the inequality*

$$g(t, x_2, y_2) - g(t, x_1, y_1) \geqslant -M(x_2 - x_1) - N(y_2 - y_1)$$

holds.

(H_3) *For* x, y *such that* $\alpha(t_k) \leqslant y \leqslant x \leqslant \beta(t_k)$, *the functions* $I_k \in C(R, R)$ *satisfy the inequalities* $I_k(x) - I_k(y) \geqslant -L_k(x - y)$ $(k = 1, 2, \ldots, p)$.

(H_4) *The constants* M, N, L_k $(k = 1, 2, \ldots, p)$ *are such that* $M > 0$, $N > 0$, $0 \leqslant L_k < 1$ *and*

$$N \int_0^T e^{M(t-\theta(t))}\, dt + \sum_{k=1}^{p} L_k \leqslant 1. \tag{4.1}$$

Then there exist two sequences of functions $\{\alpha_k\}_{k=0}^{\infty}$ *and* $\{\beta_k\}_{k=0}^{\infty}$ *which are monotone increasing and monotone decreasing*, *respectively*, *and converge uniformly in the intervals* $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p$, *and their limits are minimal and maximal solutions*, *respectively*, *in* $[\alpha, \beta]$ *of the PBVP* (1.1).

Proof. If α and β are lower and upper solutions for (1.1) respectively, then the functions $\bar{\alpha}$ and $\bar{\beta}$ defined by (3.2) and (3.3) verify that $\bar{\alpha}(0) \leqslant \bar{\alpha}(T)$, $\bar{\beta}(0) \geqslant \bar{\beta}(T)$, $\alpha(t) \leqslant \bar{\alpha}(t)$, $\bar{\beta}(t) \leqslant \beta(t)$, for $t \in J$, and $\bar{\alpha} \leqslant \bar{\beta}$. To prove the last assertion, take $m = \bar{\alpha} - \bar{\beta} \in E$. Then $m(0) = \alpha(0) - \beta(0) \leqslant 0$. In the case $\alpha(0) > \alpha(T)$ and $\beta(0) < \beta(T)$, we have, according to condition (H_2), that

$$\begin{aligned}
& m'(t) + Mm(t) + Nm(\theta(t)) \\
&\quad = \bar{\alpha}'(t) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)) - [\bar{\beta}'(t) + M\bar{\beta}(t) + N\bar{\beta}(\theta(t))] \\
&\quad = \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) + \frac{Mt + \theta(t) + 1}{T}[\alpha(0) - \alpha(T)] \\
&\qquad - [\beta'(t) + M\beta(t) + N\beta(\theta(t)) - \frac{Mt + \theta(t) + 1}{T}(\beta(T) - \beta(0))] \\
&\quad \leqslant g(t, \alpha(t), \alpha(\theta(t))) - g(t, \beta(t), \beta(\theta(t))) \\
&\qquad + M\alpha(t) + N\alpha(\theta(t)) - M\beta(t) - N\beta(\theta(t)) \leqslant 0,
\end{aligned}$$

and

$$\begin{aligned}\Delta m(t_k) &= \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k) = \Delta\alpha(t_k) - \Delta\beta(t_k)\\ &\leqslant I_k(\alpha(t_k)) - l_k - I_k(\beta(t_k)) - l_k^*\\ &\leqslant -L_k(\alpha(t_k) - \beta(t_k)) - l_k - l_k^* \leqslant -L_k m(t_k).\end{aligned}$$

The validity of these inequalities in other cases can be proved analogously. Now, using condition (H_4) and applying Lemma 2.1, we obtain that $u \leqslant 0$ on J and then $\bar{\alpha}(t) \leqslant \bar{\beta}(t)$ for $t \in J$.

Moreover, $\bar{\alpha}$ and $\bar{\beta}$ are, respectively, classical lower and upper solutions for (1.1). Indeed, if $\alpha(0) > \alpha(T)$, then

$$\begin{aligned}\bar{\alpha}'(t) &= \alpha'(t) + \frac{1}{T}(\alpha(0) - \alpha(T))\\ &\leqslant g(t, \alpha(t), \alpha(\theta(t))) - \frac{Mt + N\theta(t)}{T}(\alpha(0) - \alpha(T))\\ &\leqslant g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M(\bar{\alpha}(t) - \alpha(t))\\ &\quad + N(\bar{\alpha}(\theta(t)) - \alpha(\theta(t))) - \frac{Mt + N\theta(t)}{T}(\alpha(0) - \alpha(T))\\ &= g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))),\end{aligned}$$

and

$$\begin{aligned}\Delta\bar{\alpha}(t_k) &= \Delta\alpha(t_k) \leqslant I_k(\alpha(t_k)) - \frac{L_k t_k}{T}(\alpha(0) - \alpha(T))\\ &\leqslant I_k(\bar{\alpha}(t_k)) + L_k(\bar{\alpha}(t_k) - \alpha(t_k)) - \frac{L_k t_k}{T}(\alpha(0) - \alpha(T))\\ &\leqslant I_k(\bar{\alpha}(t_k)),\end{aligned}$$

and it is trivial when $\alpha(0) \leqslant \alpha(T)$. Thus, $\bar{\alpha}$ is a classical lower solution for (1.1). Analogously for $\bar{\beta} \in [\alpha, \beta]$ and using condition (H_2), we get, for the case $\beta(0) < \beta(T)$, that

$$\begin{aligned}\bar{\beta}'(t) &= \beta(t) - \frac{1}{T}(\beta(T) - \beta(0))\\ &\geqslant g(t, \beta(t), \beta(\theta(t))) + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0))\\ &\geqslant g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))) - M(\beta(t) - \bar{\beta}(t)) - N(\beta(\theta(t)) - \bar{\beta}(\theta(t)))\\ &\quad + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0)) = g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))) - \frac{Mt}{T}(\beta(T) - \beta(0))\\ &\quad - \frac{N\theta(t)}{T}(\beta(T) - \beta(0)) + \frac{Mt + N\theta(t)}{T}(\beta(T) - \beta(0)) = g(t, \bar{\beta}(t), \bar{\beta}(\theta(t))),\end{aligned}$$

and

$$\begin{aligned}\Delta\bar{\beta}(t_k) &= \Delta\beta(t_k) \geqslant I_k(\beta(t_k)) - \frac{L_k t_k}{T}(\beta(T) - \beta(0))\\ &\geqslant I_k(\bar{\beta}(t_k)) + L_k(\bar{\beta}(t_k) - \beta(t_k)) - \frac{L_k t_k}{T}(\beta(T) - \beta(0))\\ &\geqslant I_k(\bar{\beta}(t_k)),\end{aligned}$$

and obviously for $\beta(0) \geqslant \beta(T)$. So $\bar{\beta}$ is a classical upper solution for (1.1).

Fix a function $\eta \in [\bar{\alpha}, \bar{\beta}]$ and consider the following PBVP for the linear differential equation:

$$\begin{cases} u'(t) + Mu(t) + Nu(\theta(t)) = \sigma_\eta(t), & t \in J^-, \\ \Delta u(t_k) = -L_k u(t_k) + \gamma_k(\eta), & k = 1, 2, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{4.2}$$

where

$$\sigma_\eta(t) = g(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t))$$

and

$$\gamma_k(\eta) = I_k(\eta(t_k)) + L_k\eta(t_k), \quad k = 1, 2, \ldots, p.$$

The function $\alpha(t)$ and $\beta(t)$ are lower and upper solutions, respectively, of the PBVP (4.2). Indeed,

If $\alpha(0) \leqslant \alpha(T)$, we get

$$\begin{aligned} \alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) &\leqslant g(t, \alpha(t), \alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) \\ &\leqslant g(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t)) = \sigma_\eta(t), \end{aligned}$$

and

$$\begin{aligned} \Delta\alpha(t_k) &\leqslant I_k(\alpha(t_k)) \leqslant I_k(\eta(t)) + L_k(\eta(t_k) - \alpha(t_k)) \\ &= -L_k\alpha(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k). \end{aligned}$$

If $\alpha(0) > \alpha(T)$, then we have

$$\begin{aligned} &\alpha'(t) + M\alpha(t) + N\alpha(\theta(t)) \\ &\quad\leqslant g(t, \alpha(t), \alpha(\theta(t))) + M\alpha(t) + N\alpha(\theta(t)) - \frac{Mt + N\theta(t) + 1}{T}(\alpha(0) - \alpha(T)) \\ &\quad\leqslant g(t, \eta(t), \eta(\theta(t))) + M\eta(t) + N\eta(\theta(t)) - a(t) \\ &\quad= \sigma_\eta(t) - a(t), \end{aligned}$$

and

$$\begin{aligned} \Delta\alpha(t_k) &\leqslant I_k(\alpha(t_k)) - \frac{L_k t_k}{T}(\alpha(0) - \alpha(T)) \\ &\leqslant I_k(\eta(t)) + L_k(\eta(t_k) - \alpha(t_k)) - l_k \\ &= -L_k\alpha(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k) - l_k. \end{aligned}$$

Thus, $\alpha(t)$ is a lower solution of the PBVP (4.2). Analogously, we can prove $\beta(t)$ is a upper solution of the PBVP (4.2). And we can also prove that $\bar{\alpha}$ and $\bar{\beta}$ are classical lower and upper solutions of PBVP (4.2). Therefore we have shown that the hypotheses of Theorem 3.1 are verified, we could get that there exists a unique solution u for (4.2) and that $\bar{\alpha} \leqslant u \leqslant \bar{\beta}$. In this case, we could define $\mathscr{A} : \eta \in [\bar{\alpha}, \bar{\beta}] \to \mathscr{A}\eta = u \in [\bar{\alpha}, \bar{\beta}]$.

We complete the proof by four steps:

Step 1: We claim that $\bar{\alpha} \leqslant \mathscr{A}\bar{\alpha}$ and $\mathscr{A}\bar{\beta} \leqslant \bar{\beta}$.

Let $\alpha_1 = \mathscr{A}\bar{\alpha}$, then α_1 satisfies

$$\begin{cases} \alpha_1'(t) + M\alpha_1(t) + N\alpha_1(\theta(t)) = g(t, \bar{\alpha}(t), \bar{\alpha}(\theta(t))) + M\bar{\alpha}(t) + N\bar{\alpha}(\theta(t)), \\ \Delta\alpha_1(t_k) = -L_k\alpha_1(t_k) + I_k(\bar{\alpha}(t_k)) + L_k\bar{\alpha}(t_k), \\ \alpha_1(0) = \alpha_1(T). \end{cases}$$

Let $m(t)=\bar{\alpha}(t)-\alpha_1(t)$, and $\bar{\alpha}(t)$ is a classical lower solution of (4.2), we have

$$\begin{aligned}
&m'(t)+Mm(t)+Nm(\theta(t))\\
&\quad=\bar{\alpha}'(t)-\alpha_1(t)+M\bar{\alpha}(t)-M\alpha_1(t)+N\bar{\alpha}(\theta(t))-N\alpha_1(\theta(t))\\
&\quad\leqslant g(t,\bar{\alpha}(t),\bar{\alpha}(\theta(t)))+M\bar{\alpha}(t)+N\bar{\alpha}(\theta(t))-g(t,\bar{\alpha}(t),\bar{\alpha}(\theta(t)))+M\bar{\alpha}(t)+N\bar{\alpha}(\theta(t))\\
&\quad=0.
\end{aligned}$$

And

$$\begin{aligned}
\Delta m(t_k)&=\Delta\bar{\alpha}(t_k)-\Delta\alpha_1(t_k)\\
&\leqslant I_k(\bar{\alpha}(t_k))+L_k\alpha_1(t_k)-I_k(\bar{\alpha}(t_k))-L_k\bar{\alpha}(t_k)\\
&\leqslant -L_k(\bar{\alpha}(t_k)-\alpha_1(t_k))\leqslant -L_km(t_k),
\end{aligned}$$

with $m(0)=\bar{\alpha}(0)-\alpha_1(0)\leqslant\bar{\alpha}(T)-\alpha_1(T)=m(T)$, then by Lemma 2.2, $m(t)\leqslant 0$, which implies $\bar{\alpha}(t)\leqslant\mathscr{A}\bar{\alpha}(t)$, i.e., $\bar{\alpha}\leqslant\mathscr{A}\bar{\alpha}$. Similarly, we can prove $\mathscr{A}\bar{\beta}\leqslant\bar{\beta}$.

Step 2: We show that $\mathscr{A}\eta_1\leqslant\mathscr{A}\eta_2$ if $\bar{\alpha}\leqslant\eta_1\leqslant\eta_2\leqslant\bar{\beta}$.

Let $\eta_1^*=\mathscr{A}\eta_1$, $\eta_2^*=\mathscr{A}\eta_2$ and $m=\eta_1^*-\eta_2^*$, then for $t\neq t_k$, $t\in J$, and by condition (H_2), we obtain

$$m'(t)+Mm(t)+Nm(\theta(t))\leqslant 0.$$

$\Delta m(t_k)=-L_k\mathscr{A}\eta_1+I_k(\eta_1)+L_k\eta_1+L_k\mathscr{A}\eta_2-I_k(\eta_2)+L_k\eta_2\leqslant -L_km(t_k),\ k=1,2,\ldots,p$ (condition (H_3)).

It is easy to verify that $m(0)=m(T)$.

Still by Lemma 2.2, we obtain $m\leqslant 0$, which implies $\mathscr{A}\eta_1\leqslant\mathscr{A}\eta_2$.

Step 3: We prove that PBVP (1.1) have solutions.

Let $\alpha_n=\mathscr{A}\alpha_{n-1}$, $\beta_n=\mathscr{A}\beta_{n-1}$, $n=1,2,\ldots$. Following the first two steps, we have

$$\bar{\alpha}=\alpha_0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_n\leqslant\cdots\leqslant\beta_n\leqslant\cdots\leqslant\beta_1\leqslant\beta_0=\bar{\beta}.$$

Obviously, each $\alpha_i,\ \beta_i\ (i=1,2,\ldots)$ satisfies

$$\begin{cases}
\alpha_i'(t)+M\alpha_i(t)+N\alpha_i(\theta(t))=g(t,\alpha_{i-1}(t),\alpha_{i-1}(\theta(t)))\\
\quad+M\alpha_{i-1}(t)+N\alpha_{i-1}(\theta(t)),\quad t\neq t_k,\ \ t\in J,\\
\Delta\alpha_i(t_k)=-L_k\alpha_i(t_k)+I_k(\alpha_{i-1}(t_k))+L_k\alpha_{i-1}(t_k),\\
\quad k=1,2,\ldots,p,\\
\alpha_i(0)=\alpha_i(T)
\end{cases}$$

and

$$\begin{cases}
\beta_i'(t)+M\beta_i(t)+N\beta_i(\theta(t))=g(t,\beta_{i-1}(t),\beta_{i-1}(\theta(t)))\\
\quad+M\beta_{i-1}(t)+N\beta_{i-1}(\theta(t)),\quad t\neq t_k,\ \ t\in J\\
\Delta\beta_i(t_k)=-L_k\beta_i(t_k)+I_k(\beta_{i-1}(t_k))+L_k\beta_{i-1}(t_k),\\
\quad k=1,2,\ldots,p,\\
\beta_i(0)=\beta_i(T).
\end{cases}$$

Therefore there exist y_* and y^* such that

$$\lim_{i\to+\infty}\alpha_i(t)=y_*(t),\qquad \lim_{i\to+\infty}\beta_i(t)=y^*(t)\quad\text{uniformly on } t\in J.$$

Clearly, y_* and y^* satisfy PBVP (1.1).

Step 4: We prove y_* and y^* are extreme solutions of PBVP (1.1).

Let $y(t)$ be any solution of PBVP (1.1), which satisfies $\bar{\alpha}(t)\leqslant y(t)\leqslant\bar{\beta}(t)$, $t\in J$. Also suppose there exists a positive integer n such that for $t\in J$, $\alpha_n(t)\leqslant y(t)\leqslant\beta_n(t)$.

Setting $m(t) = \alpha_{n+1} - y(t)$, then for $t \in J$,

$$\begin{aligned} m'(t) &= \alpha'_{n+1}(t) - y'(t) \\ &= -M\alpha_{n+1}(t) - N\alpha_{n+1}(\theta(t)) + g(t, \alpha_n(t), \alpha_n(\theta(t))) \\ &\quad + M\alpha_n(t) + N\alpha_n(\theta(t)) - g(t, y(t), y(\theta(t))) \\ &= -M\alpha_{n+1}(t) - N\alpha_{n+1}(\theta(t)) + My(t) + Ny(\theta(t)) \\ &\quad + \{g(t, \alpha_n(t), \alpha_n(\theta(t))) - g(t, y(t), y(\theta(t))) + M(\alpha_n(t) - y(t)) \\ &\quad + N(\alpha_n(\theta(t)) - y(\theta(t)))\} \leqslant -Mm(t) - Nm(\theta(t)) \quad (\text{by } (\mathrm{H}_2)), \end{aligned}$$

and

$$\Delta m(t_k) = \Delta\alpha_{n+1}(t_k) - \Delta y(t_k) \leqslant -L_k m(t_k), \quad k = 1, 2, \ldots, p, \quad m(0) = m(T).$$

By Lemma 2.2, we have for all $t \in J$, $p(t) \leqslant 0$, i.e., $\alpha_{n+1}(t) \leqslant y(t)$. Similarly, we can prove $y(t) \leqslant \beta_{n+1}$, $t \in J$. Thus, $\alpha_{n+1} \leqslant y(t) \leqslant \beta_{n+1}$, for $t \in J$, which implies $y_*(t) \leqslant y(t) \leqslant y^*(t)$. We complete the proof. □

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